Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 332 (2007) 607–616
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# The zeros of Euler's Psi function and its derivatives

Peter Walker

*College of Arts and Science, American University of Sharjah, PO 26666, Sharjah, United Arab Emirates*

Received 31 August 2006
Available online 22 November 2006
Submitted by B.C. Berndt

**Abstract**

We investigate the locations of the points of inflexion of Euler's Psi function, and the positions of the stationary points of its derivative. We also establish some trigonometric approximations to Psi which lead to improved estimates for the positions of its zeros. Finally we consider the behaviour of the horizontal separation between the branches.


*Keywords:* Euler's Psi function; Location of zeros; Separation of zeros

## 1. Introduction and notation

Euler's Psi function, defined by

$$\psi(x) := -\frac{1}{x} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n}\right),$$

is meromorphic in $\mathbb{C}$ with a simple pole at which the residue is $-1$ at each integer $\leqslant 0$. Here $\gamma = \lim_{j\to\infty}(\sum_1^j 1/i - \ln j) = 0.577\ldots$ is Euler's constant. Its graph (strictly, the graph of its restriction to the real axis) consists of a number of disconnected branches; this graph is Fig. 6.2 in [1], where asymptotic estimates and approximate numerical values for the zeros of $\psi$ are given. Our first objective is to show that each branch contains a unique point of inflexion and to investigate the location of this point. These inflexions determine the minima of the derivative $\psi'$ whose positions we shall also determine. The second objective is to establish some trigonomet-

*E-mail address:* peterw@aus.edu.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2006.10.050

ric approximations to $\psi$ and $\psi'$ which may be of some independent interest, and which result in improved estimates for the zeros of $\psi$. Finally we consider the horizontal separation of the branches, showing that as a function of the ordinate $y$, each distance is strictly greater than 1 with a unique critical point.

We use $h_j$ to denote the harmonic sum $\sum_1^j 1/i$; notation for other sums will introduced as necessary.

## 2. Critical points

It turns out to be more convenient to consider the functions,

$$F_1(x) := \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right), \tag{1}$$

and for $k \geqslant 2$,

$$F_k(x) := \sum_{n=0}^{\infty}\left(\frac{1}{x-n}\right)^k = \frac{-1}{k-1}F'_{k-1}(x) = \frac{(-1)^{k-1}}{(k-1)!}F_1^{(k-1)}. \tag{2}$$

These series are used as the basis for constructing the Gamma function in [4, Chapter 2]. Since

$$F_1(x) = \psi(-x) + \gamma, \tag{3}$$

results for $F_1$ immediately imply corresponding results for $\psi$.

Let $j$ be any integer $\geqslant 0$. Since $F_1'(x) = -\sum_0^\infty 1/(x-n)^2$ is concave down on $(j, j+1)$ and tends to $-\infty$ at each end of the interval, it follows that $F_1'$ has a unique maximum on $(j, j+1)$ and hence that $F_1$ has a unique point of inflexion there. We denote this point of inflexion by $(x_j, y_j)$ and we shall show that the sequence $(x_j - j)_0^\infty$ is increasing and bounded above by $1/2$.

The following results about series will be useful.

**Lemma 1.**

(i) *Let $(a_n)_1^\infty$, $(b_n)_1^\infty$ be sequences, both tending to zero as $n \to \infty$, and let the series $\sum_1^\infty (a_n - b_n)$ be convergent with sum $S$. Then for any integer $N \geqslant 1$, the series $\sum_1^\infty (a_{n+N} - b_n)$ is convergent with sum $S - \sum_1^N a_n$.*

(ii) *Let $f$ be decreasing and concave up on $(0, \infty)$ and tend to zero at $\infty$. Then for $a > 1/2$,*

$$\int_{a-1/2}^{\infty} f(t)\,dt > \sum_{n=0}^{\infty} f(a+n) > \int_a^\infty f(t)\,dt. \tag{4}$$

**Proof.** (i) The result is trivial if the series $\sum a_n, \sum b_n$ converge, so the interest lies in the case of divergence.

We know that $\sum_1^j (a_n - b_n) \to S$ as $j \to \infty$. Then for $j > N$, the partial sum of the series $\sum_1^\infty (a_{n+N} - b_n)$ is given by

$$\sum_1^j (a_{n+N} - b_n) = \sum_1^j (a_n - b_n) - \sum_1^N a_n + \sum_{j+1}^{j+N} a_n.$$

The third sum on the right has a fixed number of terms which tend to zero individually, so this sum tends to zero and the result follows.

(ii) For the first inequality, observe that since $f$ is concave up, the graph lies above its tangent at any point, from which $f(a+n) < \int_{a+n-1/2}^{a+n+1/2} f(t)\,dt$ and the result follows by addition over $n$. The second inequality is immediate and well known. □

**Theorem 2.**

(i) $x_{j+1} - x_j > 1$,

(ii) $x_j = j + 1/2 - \eta_j$ *where*

$$\frac{1}{(j+2)^2}\left(1 - \frac{1}{192(j+1)^2}\right)^4 < 12\eta_j \sum_{n=0}^{j} (n+1/2)^{-4} < \frac{1}{(j+1)^2}, \tag{5}$$

*and in particular*

$$0 < \eta_j < \frac{1}{192(j+1)^2}, \tag{6}$$

(iii) $j^2\eta_j \to 1/(2\pi^4)$ *as* $j \to \infty$.

**Proof.** (i) Observe that $x_j$ is the unique solution in the interval $(j, j+1)$ of the equation $F_3(x) = 0$. Then $F_3(x_j) = F_3(x_{j+1}) = 0$ may be written

$$\begin{aligned}
\sum_0^\infty \frac{1}{(x_j - n)^3} = \sum_0^\infty \frac{1}{(x_{j+1} - n)^3} &= \frac{1}{x_{j+1}^3} + \sum_1^\infty \frac{1}{(x_{j+1} - n)^3} \\
&= \frac{1}{x_{j+1}^3} + \sum_0^\infty \frac{1}{(x_{j+1} - n - 1)^3}, \\
(x_{j+1} - x_j - 1)\sum_0^\infty \frac{p^2 + pq + q^2}{p^3q^3} &= \frac{1}{x_{j+1}^3},
\end{aligned}$$

where $p = x_j - n$, $q = x_{j+1} - n - 1$. In this summation the numerator is positive-definite, while $p, q$ have the same sign, namely positive if $n \leqslant j$, negative if $n \geqslant j+1$. Hence both the summation and the right-hand side are positive, so $x_{j+1} - x_j - 1$ must be positive also.

(ii) The value of $F_3(j+1/2) = \sum_0^\infty (j + 1/2 - n)^{-3} = \sum_0^j (1/2 + n)^{-3} - \sum_0^\infty (1/2+n)^{-3}$ is clearly negative, and since $F_3$ is decreasing on $(j, j+1)$ it follows that $x_j < j + 1/2$ and so $\eta_j > 0$.

To estimate $\eta_j$, write $F_3(x_j) = 0$ in the form

$$\sum_{n=0}^{2j+1} \frac{1}{(x_j - n)^3} = \sum_{n=2j+2}^{\infty} \frac{1}{(n - x_j)^3}. \tag{7}$$

Since $x_j < j + 1/2$, the summation on the right of (7) is less than

$$\sum_{n=2j+2}^{\infty} (n - j - 1/2)^{-3} < \int_{j+1}^{\infty} t^{-3}\,dt = \frac{1}{2(j+1)^2}$$

by Lemma 1. On the left of (7) we have an odd function of $x-j-1/2$ which is strictly decreasing on $(j, j+1)$, and whose gradient has at $j+1/2$ its least absolute value, namely $6\sum_{n=0}^{j}(n+1/2)^{-4}$. Combining these gives the upper estimate for $\eta_j$.

For the lower estimate, we use $x_j > j$ to find a lower estimate for the sum on the right of (7), namely

$$\sum_{n=2j+2}^{\infty}\frac{1}{(n-j)^3}=\sum_{n=j+2}^{\infty}\frac{1}{n^3}>\int_{j+2}^{\infty}t^{-3}\,dt=\frac{1}{2(j+2)^2}. \tag{8}$$

Similarly, with $x=j+1/2-t$, we write the sum on the left of (7) as

$$\begin{aligned}
S(t)&=\sum_{n=0}^{2j+1}\frac{1}{(x-n)^3}=\sum_{n=0}^{2j+1}\frac{1}{(j+1/2-t-n)^3}\\
&=\sum_{n=0}^{j}\left[\frac{1}{(1/2+n-t)^3}-\frac{1}{(t+1/2+n)^3}\right]\\
&=\sum_{n=0}^{j}\frac{1}{(n+1/2)^3}\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}t^m\frac{(-1)^m-1}{(n+1/2)^m}\\
&=-\sum_{m\text{ odd},\,\geqslant 1}^{\infty}(m+1)(m+2)t^m\sum_{n=0}^{j}\frac{1}{(n+1/2)^{m+3}}.
\end{aligned}$$

Hence

$$S'(t)=-\sum_{m\text{ odd},\geqslant 1}^{\infty}m(m+1)(m+2)t^{m-1}\sum_{n=0}^{j}\frac{1}{(n+1/2)^{m+3}},$$

$$\begin{aligned}
\left|S'(t)\right|&<\sum_{n=0}^{j}\frac{1}{(n+1/2)^4}\sum_{m\text{ odd},\geqslant 1}^{\infty}m(m+1)(m+2)t^{m-1}\\
&=3\sum_{n=0}^{j}\frac{1}{(n+1/2)^4}\left[\frac{1}{(1-t)^4}+\frac{1}{(1+t)^4}\right]\\
&<\frac{6}{(1-t)^4}\sum_{n=0}^{j}\frac{1}{(n+1/2)^4}.
\end{aligned}$$

In this estimate we substitute from (5) with $t=\eta_j<1/(192(j+1)^2)$, and combining this with (8) gives the result stated.

(iii) This follows at once from (ii) on letting $j\to\infty$, since $\sum_0^{\infty}(n+1/2)^4=\pi^4/6$. □

Recall that $(x_j, y_j)$ is the inflexion of $F_1$ on $(j, j+1)$, or equivalently $(-x_j, y_j)$ is the inflexion of $\psi$ on $(-j-1, -j)$. Let $(x_j, z_j)$ be the minimum point of $F_2$ on $(j, j+1)$, or equivalently let $(-x_j, z_j)$ be the minimum point of $\psi'$ on $(-j-1, -j)$. Also $\eta_j$ denotes $j+1/2-x_j$.

**Theorem 3.** *Let*

$$\sigma_j := \sum_{n=0}^{\infty} \frac{(-1)^n}{j+1+n/2}, \qquad \tau_j := \sum_{n=0}^{\infty} \frac{1}{(n+j+3/2)^2}, \qquad \mu_j := \sum_{n=0}^{\infty} \frac{1}{(n+j+3/2)^3}.$$

(i) *The sequence* $(y_j)$ *is increasing and satisfies*

$$h_j + \sigma_j < y_j < h_j + \sigma_j + \frac{\pi^2}{192(j+1)^2}.$$

(ii) *The sequence* $(z_j)$ *is increasing and satisfies*

$$\pi^2 - \tau_j - 2\mu_j\eta_j < z_j < \pi^2 - \tau_j.$$

**Proof.** (i) To begin with we note the values

$$\begin{aligned} F_1(j+1/2) &= \frac{1}{j+1/2} + \sum_{n=1}^{\infty}\left(\frac{1}{j+1/2-n} + \frac{1}{n}\right) \\ &= h_j + \sum_{n=0}^{\infty} \frac{(-1)^n}{j+1+n/2} \quad \text{(using Lemma 1)} \\ &= h_j + \sigma_j, \end{aligned} \tag{9}$$

$$\begin{aligned} F_2(j+1/2) &= -F_1'(j+1/2) = \sum_{n=0}^{\infty} \frac{1}{(j+1/2-n)^2} \\ &= 2\sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2} - \sum_{n=j+1}^{\infty} \frac{1}{(n+1/2)^2} \\ &= \pi^2 - \tau_j < \pi^2. \end{aligned}$$

Since $x_j < j + 1/2$, we deduce at once from (9) that $y_j = F_1(x_j) > h_j + \sigma_j$.

For the upper estimate, note that since $(x_j, y_j)$ gives the position of the inflexion of $F_1$, the graph will be concave down for $x_j < x < j + 1/2$ and so the value of $F_1(x_j)$ will be less than the value given by the tangent at $j + 1/2$. Hence $y_j < F_1(j+1/2) - \eta_j F_1'(j+1/2) < h_j + \sigma_j + \pi^2/(192(j+1)^2)$. To show that $y_j < y_{j+1}$ it is sufficient to show that

$$h_j + \sigma_j + \frac{\pi^2}{192(j+1)^2} < h_{j+1} + \sigma_{j+1}, \quad \text{or}$$

$$\frac{\pi^2}{192(j+1)^2} < \frac{1}{j+1} + \left(\frac{-1}{j+1} + \frac{1}{j+3/2}\right)$$

which is clearly true.

(ii) Since $(x_j, z_j)$ is the minimum of $F_2$ it is immediate from (9) that $z_j < F_2(j+1/2) = \pi^2 - \tau_j$. For the lower estimate, we observe that $F_2$ is concave down on $(j, j+1)$ and hence that its value at $x_j$ will be greater than the value given by the tangent at $j + 1/2$. This gives

$$\begin{aligned} z_j &> F_2(j+1/2) - \eta_j F_2'(j+1/2) \\ &= \pi^2 - \tau_j - 2\eta_j\mu_j \end{aligned}$$

as stated, and a similar argument to that in (i) gives monotonicity. □

## 3. Trigonometric approximations

**Proposition 4.** *Let*

$$s_j := \sum_{i=j}^{\infty} 1/i^2.$$

*Then for* $j < x < j+1$,

$$\pi \cot(\pi x) + h_j < F_1(x) < \pi \cot(\pi x) + h_{j+1}, \tag{10}$$

$$\pi^2 \csc^2(\pi x) - s_{j+1} < F_2(x) < \pi^2 \csc^2(\pi x) - s_{j+2}. \tag{11}$$

**Proof.** We have from (1) that

$$\begin{aligned} F_1(x) &= \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \\ &= \pi \cot(\pi x) - \sum_{n=-\infty}^{-1}\left(\frac{1}{x-n} + \frac{1}{n}\right) \\ &= \pi \cot(\pi x) - \sum_{n=1}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n}\right). \end{aligned} \tag{12}$$

Suppose that $j < x < j+1$. Since the terms in the last sum are decreasing in $x$, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{j+1+n} - \frac{1}{n}\right) < \sum_{n=1}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n}\right) < \sum_{n=1}^{\infty}\left(\frac{1}{j+n} - \frac{1}{n}\right), \quad \text{or}$$

$$-h_{j+1} < \sum_{n=1}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n}\right) < -h_j$$

using Lemma 1. Combining this with (12) gives (10).

Similarly

$$\begin{aligned} F_2(x) &= \frac{1}{x^2} + \sum_{n=1}^{\infty}\frac{1}{(x-n)^2} \\ &= \pi \csc^2(\pi x) - \sum_{n=-\infty}^{-1}\frac{1}{(x-n)^2} \\ &= \pi \csc^2(\pi x) - \sum_{n=1}^{\infty}\frac{1}{(x+n)^2}. \end{aligned} \tag{13}$$

Then as above

$$\sum_{n=1}^{\infty}\frac{1}{(j+1+n)^2} < \sum_{n=1}^{\infty}\frac{1}{(x+n)^2} < \sum_{n=1}^{\infty}\frac{1}{(j+n)^2}$$

and combining this with (13) gives (11). $\square$

This allows us to make the following estimate for the positions of the zeros of $\psi$.

**Theorem 5.** *Let $-\alpha_j$ be the zero of $\psi$ in the interval $(-j-1,-j)$. Then for $j \geqslant 1$,*

$$\frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{h_{j+1}-\gamma}\right) < j+1-\alpha_j < \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{h_j-\gamma}\right), \tag{14}$$

*where* $\tan^{-1}$ *denotes the principal branch whose value is between $-\pi/2$ and $\pi/2$.*

**Proof.** From (3) it follows that if $\psi(-\alpha_j)=0$ then $F_1(\alpha_j)=\gamma$, and so from (10),

$$\begin{aligned}
&\pi\cot(\pi\alpha_j)+h_j < \gamma < \pi\cot(\pi\alpha_j)+h_{j+1},\\
&\gamma-h_{j+1} < \pi\cot(\pi\alpha_j) < \gamma-h_j,\\
&\frac{1}{h_{j+1}-\gamma} < \pi\tan(-\pi\alpha_j) = \pi\tan\bigl(\pi(j+1-\alpha_j)\bigr) < \frac{1}{h_j-\gamma}
\end{aligned}$$

and the result follows. □

Note that the derivative of $\tan^{-1}$ is $<1$, and so the difference between the two sides of (14) is bounded above by $1/((j+1)(h_j-\gamma)^2) = O(j^{-1}(\ln j)^{-2})$, giving an estimate for $\alpha_j$ which improves on the order of magnitude of the error term in [1, Eq. 6.3.20]; the result there is that $\alpha_{n-1} = n-(\ln n)^{-1} + o[(\ln n)^{-2}]$.

## 4. Separation of branches

In this section the objective is to investigate the properties of the horizontal distance between successive branches of the graph of $F_1$. The result for $\psi$ is the same, using (3). We begin by showing that the separation of the branches is bounded below by 1. (This is the analogue for infinite sums of the result in [3], compare also [2]; its proof is repeated here simply for convenience.)

**Lemma 6.** *Let $j$ be an integer $\geqslant 0$ and let $u$, $v$ satisfy $j<u<j+1<v<j+2$ and $F_1(u)=F_1(v)$. Then $v-u>1$.*

**Proof.** We can write $F_1(u)=F_1(v)$ successively as

$$\begin{aligned}
&\frac{1}{u}+\sum_1^\infty\left(\frac{1}{u-n}+\frac{1}{n}\right) = \frac{1}{v}+\sum_1^\infty\left(\frac{1}{v-n}+\frac{1}{n}\right),\\
&\frac{1}{u}+\sum_1^\infty\left(\frac{1}{u-n}-\frac{1}{v-n}\right) = \frac{1}{v},\\
&\sum_0^\infty\left(\frac{1}{u-n}-\frac{1}{v-n-1}\right) = \frac{1}{v},\\
&(v-u-1)\sum_0^\infty\frac{1}{(u-n)(v-n-1)} = \frac{1}{v}.
\end{aligned}$$

In this last summation, the factors $u-n$ and $v-n-1$ are both positive for $n \leqslant j$ and both negative for $n \geqslant j+1$. Hence both the summation and $1/v$ are positive so $v-u-1$ is positive too. □

Theorem 2(i) is the corresponding result for the zeros of $F_1''$; evidently the same holds for all even derivatives of $F_1$.

Our main result is as follows.

**Theorem 7.** *For $y \in \mathbb{R}$, $j \geqslant 0$ let $g_j(y)$ be the unique value of $x$ in $(j, j+1)$ with $F_1(x) = y$. (Here $g_j$ is simply the inverse of the restriction of $F_1$ to $(j, j+1)$.) Then the function $d_j := g_{j+1} - g_j$, which measures the horizontal distance between the corresponding branches, is bounded below by* 1 *and is strictly increasing as $y$ increases from $-\infty$, to a unique maximum, and then strictly decreasing as $y$ increases to $+\infty$.*

**Proof.** Let $(x_j, y_j)$ denote the point of inflexion of $F_1$ which lies in $(j, j+1)$ and let $(x_j, z_j)$ denote the minimum point of $F_2$ on $(j, j+1)$. We shall need the result from Theorem 2 that the sequence $(x_j - j)_0^\infty$ is increasing and bounded above by $1/2$.

The proof is in two parts, depending on the value of $y$. For $y \geqslant y_{j+1}$, the $(j+1)$st inflexion of $F_1$, we shall show that $d_j$ is strictly decreasing from $d_j(y_{j+1})$ to 1 as $y$ increases to $\infty$; this turns out to be the major part of the proof. Having done this we show relatively easily that for $y \leqslant y_{j+1}$, that $d_j$ is strictly increasing from 1 to a unique maximum and then decreasing to $d_j(y_{j+1})$ as $y$ increases from $-\infty$ to $y_{j+1}$.

To begin with, suppose that $u, v, y$ satisfy

$$j < u < j+1 < v < j+2 \quad \text{and} \quad F_1(u) = F_1(v) = y \tag{15}$$

where we know from Lemma 6 that $u + 1 < v$.

We want to show that for $y \geqslant y_{j+1}$, $d_j(y)$ is decreasing. Since $g_j$ is inverse to $F_1$ on $(j, j+1)$ it follows that $g_j' = 1/F_1' = -1/F_2$ and hence the requirement that $d_j(y)$ is decreasing is equivalent to $F_2(u) > F_2(v)$. Also the condition $y \geqslant y_{j+1}$ is equivalent to $v \leqslant x_{j+1}$. From (12) and (13) we have successively

$$\begin{aligned} y &= F_1(u) = \pi \cot(\pi u) + F_1(-1-u), \\ F_2(u) &= \pi^2 \csc^2(\pi u) - F_2(-1-u) \\ &= \pi^2 + \pi^2 \cot^2(\pi u) - F_2(-1-u) \\ &= \pi^2 + \big(y - F_1(-1-u)\big)^2 - F_2(-1-u) \end{aligned}$$

and similarly

$$F_2(v) = \pi^2 + \big(y - F_1(-1-v)\big)^2 - F_2(-1-v).$$

Hence the required $F_2(u) > F_2(v)$ is equivalent successively to

$$\begin{aligned} &-2yF_1(-1-u) + F_1^2(-1-u) - F_2(-1-u) \\ &\quad > -2yF_1(-1-v) + F_1^2(-1-v) - F_2(-1-v), \\ &2yF_1(-1-v) - 2yF_1(-1-u) \\ &\quad > F_1^2(-1-v) - F_1^2(-1-u) + F_2(-1-u) - F_2(-1-v), \\ &\big(2y - F_1(-1-v) - F_1(-1-u)\big)\big(F_1(-1-v) - F_1(-1-u)\big) \\ &\quad > F_2(-1-u) - F_2(-1-v). \end{aligned} \tag{16}$$

We write the factors on the two sides of (16) as $A := 2y - F_1(-1-v) - F_1(-1-u)$, $B := F_1(-1-v) - F_1(-1-u)$, $C := F_2(-1-u) - F_2(-1-v)$ and show that $AB > C$. We begin with $B$. We use the functional equation $F_1(x) = 1/x + F_1(x+1)$ to obtain

$$B = F_1(-1-v) - F_1(-1-u) = F_1(-1-v) - F_1(-2-u) + \frac{1}{1+u}$$
$$= d_1 + \frac{1}{1+u} \quad \text{say,}$$

where $d_1 := F_1(-1-v) - F_1(-2-u)$ will be estimated later. Next $A = 2y - F_1(-1-v) - F_1(-1-u) = 2(y - F_1(-1-v)) + F_1(-1-v) - F_1(-1-u) = 2(y - F_1(-1-v)) + B$. But here $y \geqslant y_{j+1} > F_1(j+3/2) = F_1(-j-5/2) \geqslant F_1(-1-v)$ since $F_1$ is decreasing and $v \leqslant x_{j+1} < j+3/2$, so $-j-5/2 < -1-v$. Hence $A > B$.

Finally we use the functional equation $F_2(x) = 1/x^2 + F_2(x+1)$ to obtain

$$C = F_2(-1-u) - F_2(-1-v) = \frac{1}{(1+u)^2} + F_2(-2-u) - F_2(-1-v)$$
$$= \frac{1}{(1+u)^2} + d_2 \quad \text{say,}$$

where $d_2 := F_2(-2-u) - F_2(-1-v)$ will also be estimated later. Thus the required result that $AB > C$ will be proved if we can show that

$$B^2 = \left(d_1 + \frac{1}{1+u}\right)^2 > C = \frac{1}{(1+u)^2} + d_2$$

for which

$$\frac{2d_1}{1+u} > d_2 \tag{17}$$

is obviously sufficient.

To estimate $d_1$ and $d_2$ we proceed as follows. Recall that $u+1 < v$ and so $-1-v < -2-u$. Hence from the mean value theorem, $d_1 = F_1(-1-v) - F_1(-2-u) = (v-u-1)|F_1'(\theta)|$ for some $\theta \in (-1-v, -2-u)$ and so $F_1(-1-v) - F_1(-2-u) > (v-u-1)F_2(-1-v)$ since $|F_1'| = F_2$ is increasing on $(-\infty, 0)$. But for $a > 0$, $F_2(-a) = \sum_{n=0}^{\infty}(a+n)^{-2} > \int_a^{\infty} t^{-2}\,dt = 1/a$, and so

$$d_1 > \frac{v-u-1}{1+v}.$$

Similarly $d_2 = F_2(-2-u) - F_1(-1-v) = (v-u-1)F_2'(\phi)$ for some $\phi \in (-1-v, -2-u)$ which is less than $(v-u-1)F_2'(-2-u)$ since $F_2'$ is increasing on $(0, \infty)$. But for $a > 0$, $F_2'(-a) = 2\sum_{n=0}^{\infty}(a+n)^{-3} < 2\int_{a-1/2}^{\infty} t^{-3}\,dt = 1/(a-1/2)^2$, using Lemma 1 and so

$$d_2 < \frac{v-u-1}{(u+3/2)^2}.$$

Hence for (17) it is enough to show that

$$\frac{2(v-u-1)}{(1+v)(1+u)} > \frac{v-u-1}{(u+3/2)^2}$$

or $2(u+3/2)^2 > (1+v)(1+u)$, equivalently

$$2u^2 + 5u + 7/2 > v(1+u). \tag{18}$$

But $u > j$ so the left side is greater than $2j^2 + 5j + 7/2$, while $u+1 < v \leqslant j+3/2$ so the right side is less than $(j+3/2)^2$ which is clearly less than $2j^2 + 5j + 7/2$. This establishes (18), then (17) and so the first part of the theorem.

The second part of the proof is less strenuous, and results from considering the relative positions of the graph of $F_2(x)$ and $F_2(x+1)$.

We begin by noting that since $F_2(x+1) = F_2(x) + 1/(x+1)^2 > F_2(x)$ it follows that $z_{j+1}$, the minimum value of $F_2$ on $(j+1, j+2)$, is strictly greater than $z_j$, the minimum value of $F_2$ on $(j, j+1)$.

We consider what happens when in (15) $y$ decreases from $+\infty$. Since $F_1$ is decreasing, both $u$ and $v$ are increasing, and the first part of the proof has shown that when $y \geqslant y_{j+1}$ (equivalently when $v \leqslant x_{j+1}$), we have $F_2(u) > F_2(v) \geqslant F_2(x_{j+1}) = z_{j+1}$ which we know to be greater than $z_j = F_2(x_j)$. Hence $y$ must decrease beyond $y_{j+1}$ to reach $y_j$ where $F_2(u) = F_2(x_j)$ and we have proved that $y_{j+1} > y_j$. (This has proved the monotonicity of $(y_j)$ by a quite different route from that used in Theorem 3(i).)

Let $u_0$ be the value of $u$ which corresponds to $v = x_{j+1}$ and let $v_1$ be the value of $v$ which corresponds to $u = x_j$.

As $y$ decreases from $y_{j+1}$ to $y_j$, $u$ increases from $u_0$ to $x_j$ and $F_2(u)$ decreases from $F_2(u_0) > z_{j+1}$ to $F_2(x_j) = z_j < z_{j+1}$. At the same time $v$ increases from $x_{j+1}$ to $v_1$ and $F_2(v)$ increases from $z_{j+1}$ to $F_2(v_1)$. Hence in this interval there is exactly one point where $F_2(u) - F_2(v)$ changes from positive to negative, giving a maximum value of $d_j$.

Finally when $y$ decreases from $y_j$ to $-\infty$ we have $F_2(u) = F_2(u+1) - 1/(u+1)^2 < F_2(u+1) < F_2(v)$ since $u+1 < v$ and $F_2$ is increasing for $v \geqslant x_{j+1}$. It follows that $d_j$ decreases when $y$ decreases from $y_j$ to $-\infty$ which completes the proof. $\square$

## References


[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Appl. Math. Ser., vol. 55, National Bureau of Standards, US Government Printing Office, Washington, DC, USA, 1964.

[2] A. Stoyanoff, Sur un théorème de M. Marcel Riesz, Nouv. Ann. Math. 6 (1926) 97–99.

[3] P.L. Walker, Separation of the zeros of polynomials, Amer. Math. Monthly (1991) 272–273.

[4] P.L. Walker, Elliptic Functions, a Constructive Approach, J. Wiley and Sons, Chichester, UK, 1996.